International Journal of Theoretical Physics, Vol. 40, No. 11, November 2001 (© 2001)

Covariant Poisson Brackets of Classical Fields

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Received March 26, 2001; accepted May 13, 2001

The covariant Poisson brackets of classical fields are defined in terms of the fields covariant canonical variables. These are then consistent with the causality principle and the quantum fields covariant commutation relations.

Poisson brackets are the common ground of classical and quantum mechanics. They are used in field theories at the prequantization level. However, the brackets under disposal for the fields and conjugate momenta are noncovariant. This situation awaits a resolution (Mashkour, M. A. (1998). *International Journal of Theoretical Physics* **37**, 785). We formulate here the covariant Poisson brackets for fields in terms of the fields covariant canonical variables. These brackets are then consistent with the causality principle and the quantum fields covariant commutation relations.

1. PRELIMINARIES

1.1. The Covariant Commutation Relations

The quantum field covariant commutation relations shall be the guide for the development of the classical fields covariants Poisson brackets. We exclude the case of spinor fields. Let for the moment $\{q^\alpha(\mathrm{x}, t); \alpha = 1, 2, \ldots\}$ be a canonical set of fields that satisfy the free fields Lagrangian density:

$$L = \frac{1}{2}[\partial^\mu q^\alpha \partial_\mu q^\alpha + m_\alpha q^\alpha q^\alpha] \tag{1}$$

(summation over repeated indices is implied, unless stated otherwise.) We recognize

$$\begin{aligned} p_\alpha^\mu &= \frac{\partial L}{\partial(\partial_\mu q^\alpha)} \\ &= \partial^\mu q^\alpha \end{aligned} \tag{2}$$

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0020-7748/01/1100-2011$19.50/0

is the 4-conjugate momentum of q^α (Mashkour, 1998). The covariant commutation relations of the quantized fields $\hat{q}^\alpha$ are given as (Bjorken, 1964):

$$[\hat{q}^\alpha(\mathbf{x},t), \hat{q}^\beta(\mathbf{x}',t')] = \delta^{\alpha\beta} \cdot \int_{-\infty}^{\infty} \frac{1}{2\omega} \left(e^{-i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} - e^{i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right) d\mathbf{k} \tag{3}$$

To facilitate the correspondence with the classical canonical description, we divide the space-time into causal 4-cells of dimensions Δx^μ; with $\Delta x^\mu \Delta x_\mu = 0$. At the end we let $\Delta x^\mu \to 0$. The coarse-graining of the space-time amounts to contracting the limits of integrations in Eq. (3) as:

$$[\hat{q}^\alpha(\mathbf{x},t), \hat{q}^\beta(\mathbf{x}',t')] = \delta^{\alpha\beta} \cdot \int_{-\mathbf{K}/2}^{\mathbf{K}/2} \frac{1}{2\omega} \left(e^{-i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} - e^{i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right) d\mathbf{k} \tag{4a}$$

$$\equiv -i \cdot \delta^{\alpha\beta} \cdot \Delta(x,x') \tag{4b}$$

where

$$K_{(j)}\Delta x^{(j)} \simeq 1; \qquad j = 1, 2, 3 \tag{5}$$

(no summation over repeated indices).

Consider a pair of points $\zeta = (\boldsymbol{\zeta},\zeta^0)$, and $\zeta' = (\boldsymbol{\zeta}',\zeta^{0\prime})$ in one cell, where

$$\delta\zeta^\mu = \zeta^\mu - \zeta'^\mu; \quad |\delta\zeta^\mu| \leq \Delta x^\mu; \quad \mu = 0, 1, 2, 3 \tag{6}$$

and define the 4-function

$$f_\nu(\zeta,\zeta') = \partial_{\zeta^\nu}\Delta(\zeta,\zeta') \tag{7}$$

One can then see that

$$f_\nu(\zeta,\zeta') = O\left(\frac{1}{\Delta v}\right) \tag{8a}$$

$$\Delta(\zeta,\zeta') \sim 0 \tag{8b}$$

where Δv is the spatial volume of the cell. It is also seen that

$$[\hat{q}^\alpha(\zeta), \hat{q}^\beta(\zeta')] \sim 0 \tag{9a}$$

$$\left[\hat{p}^\nu_\alpha(\zeta), \hat{q}^\beta(\zeta')\right] = -i\delta^\beta_\alpha \cdot f^\nu(\zeta,\zeta') \tag{9b}$$

$$\left[\hat{p}^\mu_\alpha(\zeta), \hat{p}^\nu_\beta(\zeta')\right] \sim 0 \tag{9c}$$

1.2. The Covariant Canonical Formalism

The covariant canonical theory of classical fields is given in (Mashkour, 1998). A 4-generating function $F^{\mu}(q(x), q'(x))$ is introduced, such that

$$p_{\alpha}^{\mu}(x) = \frac{\partial F^{\mu}(q(x), q'(x))}{\partial q^{\alpha}(x)} \tag{10a}$$

$$p_{\alpha}^{\prime\mu}(x) = -\frac{\partial F^{\mu}(q(x), q'(x))}{\partial q^{\alpha\prime}(x)} \tag{10b}$$

where $q^{\alpha}(x)$, $p_{\alpha}^{\mu}(x)$ and $q'^{\alpha}(x)$, $p_{\alpha}'^{\mu}(x); \alpha = 1, 2, \ldots; \mu = 0, 1, 2, 3$, are the original and transformed covariant canonical variables, respectively.

It has been further realized in the framework of the covariant canonical theory (Mashkour, 1998) that $\{q^{\alpha}(x), p_{\alpha}^{\mu}(x); \alpha = 1, 2, \ldots; \mu = 0, 1, 2, 3\}$ on a space-like sheet $\sum$ are independent canonical variables. This conclusion together with Eqs. (10a,b) lead to (Goldstein, 1980; Mashkour, 1998)

$$\left\{p_{\alpha}'^{\mu}(x), q'^{\beta}(x)\right\}_{\nu} \equiv \sum_{\gamma} \left(\frac{\delta p_{\alpha}'^{\mu}(x)}{\delta p_{\gamma}^{\nu}(x)} \frac{\delta q'^{\beta}(x)}{\delta q^{\gamma}(x)} - \frac{\delta q'^{\beta}(x)}{\delta p_{\gamma}^{\nu}(x)} \frac{\delta p_{\alpha}'^{\mu}(x)}{\delta q^{\gamma}(x)} \right) \tag{11a}$$

$$= \delta_{\alpha}^{\beta} \delta_{\nu}^{\mu} \tag{11b}$$

$$\{q'^{\alpha}(x), q'^{\beta}(x)\}_{\nu} = 0 \tag{11c}$$

$$\left\{p_{\alpha}'^{\mu}(x), p_{\beta}'^{\eta}\right\}_{\nu} = 0 \tag{11d}$$

In which $\{q'^{\alpha}(x), p_{\alpha}'^{\mu}(x); \alpha = 1, 2, \ldots; \mu = 0, 1, 2, 3\}$ is an arbitrary set of covariant canonical variables.

1.3. Invariance Property

We verify that Eqs. (9) are invariant under the covariant canonical transformation. Thus, let $\{q'^{\alpha}(x), p_{\alpha}'^{\mu}(x); \alpha = 1, 2, \ldots; \mu = 0, 1, 2, 3\}$ be a set of canonical variables that are related to the variables in Eqs. (9) by a covariant canonical transformation. So that

$$q'^{\alpha}(x) = \overline{q}'^{\alpha}(q(x), p(x)) \tag{12a}$$

$$p_{\alpha}'^{\mu}(x) = \overline{p}_{\alpha}'^{\mu}(q(x), p^{\mu}(x)) \tag{12b}$$

(no summation over μ in Eq. (12b). Let (ζ, ζ') be a pair of points within one cell. Then we have that

$$\left[\hat{p}_{\alpha}'^{\mu}(\zeta), \hat{q}'^{\beta}(\zeta')\right] = \left(\frac{\delta \hat{p}_{\alpha}'^{\mu}(\zeta)}{\delta \hat{p}_{\gamma}^{\nu}(\zeta)} \frac{\delta \hat{q}'^{\beta}(\zeta')}{\delta \hat{q}^{\eta}(\zeta')} - \frac{\delta \hat{q}'^{\beta}(\zeta')}{\delta \hat{p}_{\gamma}^{\nu}(\zeta')} \frac{\delta \hat{p}_{\alpha}'^{\mu}(\zeta)}{\delta \hat{q}^{\eta}(\zeta)} \right) \times \left[\hat{p}_{\gamma}^{\nu}(\zeta), \hat{q}^{\eta}(\zeta')\right] + v \cdot t \tag{13a}$$

$$= -i\left(\frac{\delta\hat{p}'^{\mu}_{\alpha}(\zeta)}{\delta\hat{p}^{\nu}_{\gamma}(\zeta)}\frac{\delta\hat{q}'^{\beta}(\zeta')}{\delta\hat{q}^{\eta}(\zeta')} - \frac{\delta\hat{q}'^{\beta}(\zeta')}{\delta\hat{p}^{\nu}_{\gamma}(\zeta')}\frac{\delta\hat{p}'^{\mu}_{\alpha}(\zeta)}{\delta\hat{q}^{\eta}(\zeta)}\right)\cdot\delta^{\eta}_{\gamma}\cdot f^{\nu}(\zeta,\zeta') \tag{13b}$$

$$= -i\left(\frac{\delta\hat{p}'^{\mu}_{\alpha}(\zeta)}{\delta\hat{p}^{\nu}_{\gamma}(\zeta)}\frac{\delta\hat{q}'^{\beta}(\zeta')}{\delta\hat{q}^{\gamma}(\zeta')} - \frac{\delta\hat{q}'^{\beta}(\zeta')}{\delta\hat{p}^{\nu}_{\gamma}(\zeta')}\frac{\delta\hat{p}'^{\mu}_{\alpha}(\zeta)}{\delta\hat{q}^{\gamma}(\zeta)}\right)\cdot f^{\nu}(\zeta,\zeta') \tag{13c}$$

$$= -i\delta^{\beta}_{\alpha}\cdot\delta^{\mu}_{\nu}\cdot f^{\nu}(\zeta,\zeta') \tag{13d}$$

$$= -i\delta^{\beta}_{\alpha}\cdot f^{\mu}(\zeta,\zeta') \tag{13e}$$

where $v\cdot t$ in Eq. (13a) are, in view of Eqs. (9), vanishing terms. Similarly we establish the rest of Eqs. (9) for the variables $(q'^{\alpha}, p'^{\mu}_{\alpha})$. Observe in particular that the same 4-function $f^{\nu}(\zeta,\zeta')$ in Eqs. (9) applies to all possible sets of canonical variables.

2. THE COVARIANT POISSON BRACKETS

We are now in the position to write the covariant Poisson brackets for the field variables. As in above, we divide the space-time into causal 4-cells $\Delta\tau(i) = \prod_{\mu=0}^{3}\Delta x^{\mu}(i)$ with $\Delta x_{\mu}(i)\Delta x^{\mu}(i) = 0$. Afterwards, we let the dimensions $\Delta x^{\mu}(i) \to 0$. Let $\sum$ be a space-like sheet, and let $C(\sum)$ be the totality of the cells centered at $\sum$. For each cell $c(i)\varepsilon C(\sum)$, we define a pair of points $(\zeta(i), \zeta'(i))$ with the property that

$$\delta\zeta^{\mu} \equiv \zeta^{\mu}(i) - \zeta'^{\mu}(i) \tag{14}$$

is the same for all cells. We also introduce a pair of dummy points $(\zeta^{\mu},\zeta'^{\mu})$, such that

$$\zeta^{\mu} - \zeta'^{\mu} = \delta\zeta^{\mu} \tag{15}$$

in which $\delta\zeta^{\mu}$ is the same constant displacement as in Eq. (14).

We have that $(q^{\alpha}, p^{\mu}_{\alpha})$ belonging to different cells of $C(\sum)$ are independent canonical variables in the framework of the covariant canonical theory (Mashkour, 1998). This property combined with Eqs. (11) brings about

$$\sum_{k}\sum_{\gamma}\left(\frac{\delta p'^{\mu}_{\alpha}(\zeta(i))}{\delta p^{\nu}_{\gamma}(\zeta(k))}\frac{\delta q'^{\beta}(\zeta'(j))}{\delta q^{\gamma}(\zeta'(k))} - \frac{\delta q'^{\beta}(\zeta(i))}{\delta p^{\nu}_{\gamma}(\zeta(k))}\frac{\delta p'^{\mu}_{\alpha}(\zeta'(j))}{\delta q^{\gamma}(\zeta'(k))}\right) = \delta^{\beta}_{\alpha}\delta^{\mu}_{\nu}\delta^{i}_{j} \tag{16a}$$

$$\sum_{k}\sum_{\gamma}\left(\frac{\delta q'^{\alpha}(\zeta(i))}{\delta p^{\nu}_{\gamma}(\zeta(k))}\frac{\delta q'^{\beta}(\zeta'(j))}{\delta q^{\gamma}(\zeta'(k))} - \frac{\delta q'^{\beta}(\zeta(i))}{\delta p^{\nu}_{\gamma}(\zeta(k))}\frac{\delta q'^{\alpha}(\zeta'(j))}{\delta q^{\gamma}(\zeta'(k))}\right) = 0 \tag{16b}$$

$$\sum_{k}\sum_{\gamma}\left(\frac{\delta p'^{\mu}_{\alpha}(\zeta(i))}{\delta p^{\nu}_{\gamma}(\zeta(k))}\frac{\delta p'^{\nu}_{\beta}(\zeta'(j))}{\delta q^{\gamma}(\zeta'(k))} - \frac{\delta p'^{\nu}_{\beta}(\zeta(i))}{\delta p^{\nu}_{\gamma}(\zeta(k))}\frac{\delta p'^{\mu}_{\alpha}(\zeta'(j))}{\delta q^{\gamma}(\zeta'(k))}\right) = 0 \tag{16c}$$

Now, each of the sets $(q^{\alpha}(\zeta(i)), p_{\alpha}^{\mu}(\zeta(i)))$, $(q^{\alpha}(\zeta'(i)), p_{\alpha}^{\mu}(\zeta'(i)))$ is a complete data. Therefore we can express an arbitrary pair of field variables $\chi(x)$ and $\chi'(x')$ at two points x and x' as

$$\chi(x) = \bar{\chi} \mid x; q^{\alpha}(\zeta(i)),\ p_{\alpha}^{\mu}(\zeta(i))] \tag{17a}$$

$$\chi'(x') = \bar{\chi}' \mid x'; q^{\alpha}(\zeta'(i)), p_{\alpha}^{\mu}(\zeta'(i))] \tag{17b}$$

The covariant Poisson bracket of $\chi(x)$ and $\chi'(x')$ is then defined as

$$\{\chi(x), \chi'(x')\} = \sum_{\mu}\sum_{\gamma}\sum_{k}\left(\frac{\delta\bar{\chi}}{\delta p_{\gamma}^{\mu}(\zeta(k))}\frac{\delta\bar{\chi}'}{\delta q^{\gamma}(\zeta'(k))} - \frac{\delta\bar{\chi}'}{\delta p_{\gamma}^{\mu}(\zeta'(k))}\frac{\delta\bar{\chi}}{\delta q^{\gamma}(\zeta(k))}\right)\cdot f^{\mu}(\zeta,\zeta') \tag{18}$$

where $\Delta x^{\mu}(i) \to 0$. We demonstrate below that the so defined covariant Poisson bracket agrees with the covariant commutation of the quantum fields: $\hat{\chi}(x), \hat{\chi}'(x')$.

3. CONCLUDING REMARKS

3.1. Summary of the Formalism

(i) The space-time is divided into infinitesimal causal cells of dimensions Δx^{μ}. (ii) A 4-function $f_{\nu}(\zeta,\zeta')$ has been introduced by Eq. (7), where (ζ,ζ') are contained in a single cell. (iii) Equations (9) are invariant under the covariant canonical transformation, and that the same 4-function $f_{\nu}(\zeta,\zeta')$ applies to all sets of canonical variables. (iv) The fields variables throughout the cells on a space-like sheet $\sum$ are canonically independent. (v) The covariant Poisson bracket of fields was finally given by Eq. (18).

3.2. Invariances

The bracket defined by Eq. (18) is invariant under Lorentz transformation. Furthermore, according to Eqs. (16), the bracket is also invariant under the covariant canonical transformation of the basis canonical variables $(q^{\gamma}(i), p_{\gamma}^{\mu}(i))$.

3.3. The Equal-Time Poisson Brackets

Equation (18) combined with Eqs. (7), (8), and (16) leads to the standard equal-time Poisson brackets of classical fields (Mashkour, 1998):

$$\left\{p_{\alpha}^{0}(\mathbf{x}, t), q^{\beta}(\mathbf{x}', t)\right\} = \delta^{\alpha\beta}\cdot\delta(\mathbf{x}-\mathbf{x}') \tag{19a}$$

$$\{q^{\alpha}(\mathbf{x}, t), q^{\beta}(\mathbf{x}', t)\} = 0 \tag{19b}$$

$$\left\{p_{\alpha}^{0}(\mathbf{x}, t), p_{\beta}^{0}(\mathbf{x}', t)\right\} = 0 \tag{19c}$$

3.4. The Asymptotic Property

Consider the free fields q^α of Eq. (1). We determine the covariant Poisson bracket $\{q^\alpha(x), q^\beta(x')\}$, such that $\delta x^\mu = x^\mu - x'^\mu$ are infinitesimal quantities, while $|\delta x^\mu| \geq \Delta x^\mu$ (the dimension of a cell). We choose a cell (l) that contains the point x'^μ, and define

$$\zeta'^\mu(l) = x'^\mu \tag{20}$$

Let $\zeta^\mu(l) = \zeta'^\mu(l) + \delta\zeta^\mu$, where $|\delta\zeta^\mu| < \Delta x^\mu$. Then

$$x^\mu = \zeta^\mu(l) + (\delta x^\mu - \delta\zeta^\mu) \tag{21a}$$

$$q^\alpha(x) \sim q^\alpha(\zeta^\mu(l)) + (\delta x^\nu - \delta\zeta^\nu) \cdot \partial_{x^\nu} q^\alpha(\zeta(l)) \tag{21b}$$

$$= q^\alpha(\zeta^\mu(l)) + (\delta x^\nu - \delta\zeta^\nu) \cdot p_{\nu\alpha} \tag{21c}$$

Substituting into Eq. (18) we conclude, according to Eq. (7), that

$$\{q^\alpha(x), q^\beta(x')\} = \delta^{\alpha\beta} \cdot (\delta x^\nu - \delta\zeta^\nu) \cdot f_\nu(\zeta, \zeta') \tag{22a}$$

$$\sim \delta^{\alpha\beta} \cdot \Delta(x, x') \tag{22b}$$

where ζ and ζ' satisfy Eq. (15), while from Eq. (8b) $\Delta(\zeta, \zeta') \sim 0$. The Poisson bracket in Eq. (22b) corresponds faithfully with the covariant commutation relation in Eq. (4b).

3.5. The Causality Principle

Consider a pair of field variables $\chi(x)$, $\chi'(x')$ with $|(x^\mu - x'^\mu)(x_\mu - x'_\mu)| \gg 0$. The quantities $\chi(x)$, $\chi'(x')$ must, according to Eq. (17), be developments from distinguished portions on the space-like sheet $\sum$. We conclude therefore according to Eqs. (16), (17), and (18) that

$$\{\chi(x), \chi'(x')\} = 0 \tag{23}$$

which agrees with the causality principle.

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